A Study on the Relationship Between Minimal Generating Sets and Independence in Semigroups

Michael Nsikan John,
Department of Mathematics,
Akwa Ibom State University, Ikot Akpaden, Nigeria.
storm4help1@gmail.com

Sampson, Marshal Imeh,
Department of Mathematics,
Akwa Ibom State University, Ikot Akpaden, Nigeria.
marshalsampson@aksu.edu.ng.

C.F. Igiri†,
Department of Mathematics, University of Calabar,
Calabar, Cross River State, Nigeria.

O. G. Udoaka,
Department of Mathematics,
Akwa Ibom State University, Ikot Akpaden, Nigeria.

L.E. Effiong
Department of Mathematics,
Abia State Polytechnic, Aba, Nigeria.

Jackson Ante
Department of Mathematics
Top Faith University, Mkpatak, Akwa Ibom State

DOI: 10.56201/ijcsmt.v10.no1.2024.pg37.47

Abstract

This paperreviews the work ofM.I. Sampson et. Al (2023) [7], and delves into the intricate relationship between minimal generating sets and independence in semigroups by examining the comparability of elements induced by orderings on the semigroup. It demonstrates that the existence of a minimal generating set implies independence, and conversely, independence implies the existence of a minimal generating set. Additionally, the paper presents two new algorithms: one for determining minimal generating sets for countable systems of semigroups and another for any given semigroup. These algorithms offer practical solutions for semigroup theorists and *researchers. Through rigorous mathematical analysis and proof, this paper sheds light on the fundamental properties of semigroups and their generating sets.*

Keywords: *Semigroups, Independence, Minimal Generating Sets, Orderings, Algorithms*

1. INTRODUCTION

Semigroup theory is a fundamental area of algebra with wide-ranging applications in various fields. Understanding the concept of independence in semigroups and its relationship with minimal generating sets is essential for advancing our knowledge in this domain. [5] delve into the relationship between minimal generating sets and independence in semigroups, laying the foundation for further exploration of this topic. Their work serves as the basis for the current paper's investigation. Howie's book [2] provides a comprehensive overview of semigroup theory, covering key concepts, definitions, and properties. It serves as a valuable resource for understanding the fundamental principles underlying semigroups.Lallement's text [3] explores the applications of semigroup theory in combinatorics, offering insights into the practical significance of semigroups beyond abstract algebraic structures. It provides context for the theoretical aspects discussed in the current paper.Petrich's book [4] focuses on inverse semigroups, a special class of semigroups with applications in areas such as automata theory and formal languages. While not directly related to minimal generating sets, it offers additional perspectives on semigroup theory.In [6], Scheiblich'sintroduction to semigroup structure theory provides insights into the classification and characterization of semigroups. It offers a deeper understanding of the structural properties explored in the current paper. Araújo and Kochloukov [1] investigate minimal generating sets of semigroups, contributing to the understanding of generating sets and their properties. Their research complements the current paper's focus on minimal generating sets in relation to independence.

In this paper, we reviewed, improved upon the work in [7] and investigate how the comparability of elements in semigroups, induced by specific orderings, affects the concepts of minimal generating sets and independence. By exploring these relationships, we aim to provide insights into the fundamental properties of semigroups and offer practical algorithms for determining minimal generating sets in different contexts. [11] – [13] Also illustrate some of the generating sets and their ranks

2. PRELIMINARIES

Definition 2.1(Semigroup)**.** A semigroup is a set S together with a binary operation * (that is a function $*: S \times S \to S$) that satisfies the

associative property: $\forall a, b, c \in S$ such that

$$a * (b * c) = (a * b) * c$$

holds.

Illustration 2.2(Semigroup)**.** Consider the set of non-negative integers $N_0 = \{0, 1, 2, 3, \dots\}$ with the operation of addition +. Here, $(S, +)$ forms a semigroup because addition is associative. For any $a, b, c \in N_0$, we have:

$$(a + b) + c = a + (b + c)$$

Therefore, $(N_0, +)$ is a semigroup.

Definition 2.3 (Minimal Generating Set[10])**.**

Let S be a semigroup and let $H \subset S$ be a nonempty subset of S. Then H is a generating set of S if$S$

$$= \langle H \rangle := \left\{ \prod_{j=1}^{k} a_j \mid \forall \{a_j\}_{j=1}^{k} \subset H, 1 \leq k < \infty \right\},$$

where $\langle H \rangle$ is called the subsemigroup generated by H.

Illustration2.4 (Minimal Generating Set[10])**.**

(i) **Generating Sets of $\mathbb{Z}$ and $\mathbb{Q}$** (Halbeisen and Co, 2007).

Obviously, $\{1\}$ is a minimal generating set of the ring of integers $\mathbb{Z}$. On the other hand, there are also generating sets of $\mathbb{Z}$ like $\{2, 3\}$ which are not smallest. Moreover, for any set of mutually different prime numbers $p_1, \dots, p_m$, the set $\left\{\frac{n}{p_1}, \dots, \frac{n}{p_n}\right\}$ where $n = \prod_{i=1}^{m} p_i$ is a minimal generating set of $\mathbb{Z}$. (See also Remark 2.5. for an important observation about this example.)

Moreover, $\{1\}$ is the minimal generating set of $\mathbb{Q}$ $(\langle\{1\}\rangle = \mathbb{Q})$ since we can obtain any non-zero rational number from 1 by addition and division with as the field operations. Notice also that $\langle\{0,1\}\rangle = \mathbb{Q}$ but $\{0,1\}$ is not the minimal generator of $\mathbb{Q}$since we can obtain any non-zero rational numberfrom 1 by addition and division as the field operations. Notice also that $\langle\{0,1\}\rangle = \mathbb{Q}$.

Remark 2.5 (Important Observation and Questions).

From the 2.4 (i), it is clear that the additive group Z has infinite different generating sets (for example $\{1\}$ and $\{2, 3\}$) with different number of elements, hence the generating set $\{1\}$ has the smallest number of elements. The comparability of the generating subsets gives us a clearer understanding of whether to call this the minimal or minimum generating set.

Definition 2.6 (Independent Element of a Semigroup)

An element $a \in S$ is independent from a subset $U \subset S$ if $a \notin \langle U \rangle$**.**

Definition 2.7 (Independent Subsets of a Semigroup[10])

Let S be a semigroup and let $U \subset S$. If $\forall u \in U$, $u \notin \langle U \backslash \{u\} \rangle$, then U is an independent set.

Definition 2.8 (Dependent Sets of Semigroup[10])

Let S be a finite semigroup and let $U \subset S$. U is dependent if, $\exists u \in U$, $u \in \langle U \backslash \{u\} \rangle$.

Definition 2.9.(Maximal Independent Sets of semigroup).

A Maximal Independent subset $U \in S$ of a semigroup is that subset $U \in S$ in which including any otherelement in the set makes it dependent.

Illustration 2.10 (Independence and maximal independent sets[10]**).**For illustration we use this example from [8]section 1.6 of [10]by Sampson below.

Consider the six elements semigroup S, which is a 2×2 real matrix under matrix multiplication,

$$o = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; ab = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; ba = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiplication table:

	o	***i***	***a***	***b***	***ab***	***ba***
o	*o*	*o*	*o*	*o*	*o*	*o*
i	*o*	*i*	*a*	*b*	*ab*	*ba*
a	*o*	*a*	*o*	*ab*	*o*	*a*
b	*o*	*b*	*ba*	*o*	*b*	*o*
ab	*o*	*ab*	*a*	*o*	*ab*	*o*
ba	*0*	*ba*	*o*	*b*	*o*	*ba*

Let $U = \{i, a, b\} \subseteq S$. We show that U is a maximal independent subset:

If we take out i from U, notice that $\{i\}$ is excluded from the subsemigroup generated by the remaining elements $U \setminus \{i\}$ of S. That is, $\{i\}$ is excluded from $\langle U \setminus \{i\} \rangle = \{o, a, b, ab, ba\}$.

Notice also that $\{a\}$ *and* $\{b\}$ are excluded from the subsemigroup $\langle U \setminus \{a\} \rangle = \{o, i, b\}$ and $\langle U \setminus \{b\} \rangle = \{o, i, a\}$ respectively. We conclude that the subset U is independent.

Next, we show that U is a maximal independent subset of the semigroup S.

Notice that $U = \{i, a, b\}$ becomes dependent if we include any of o *or* ab *or* ba in U:

The subsemigroup generated by $\{o, i, a, b\} \setminus \{o\}$ is $\{o, i, a, b, ab, ba\}$

The subsemigroup generated by $\{ab, i, a, b\} \setminus \{ab\}$ is $\{o, i, a, b, ab, ba\}$

The subsemigroup generated by $\{ba, i, a, b\} \setminus \{ba\}$ is $\{o, i, a, b, ab, ba\}$

For $\{o, i, a, b\}$: $\{o\}$ is in the subsemigroup generated by $\{o, i, a, b\} \setminus \{o\}$

For $\{ab, i, a, b\}$: $\{ab\}$ is in the subsemigroup generated by $\{ab, i, a, b\} \setminus \{ab\}$

For $\{ba, i, a, b\}$: $\{ba\}$ is in the subsemigroup generated by $\{ba, i, a, b\} \setminus \{ba\}$

Therefore U is a maximal independent subset of S. There may exist other Maximal Independent subsets in S but the number of elements in each must be the same. The rank of S is 3 - the number of elements in U. Each maximal independent subset is called a basis of $\langle U \rangle = \{o, i, a, b, ab, ba\} = S$.

3. CENTRAL IDEA

We explore how the orderings induced by semigroup structures influence these concepts and demonstrate their interplay through mathematical analysis. Our central idea revolves around proving that minimal generating sets and independence are closely related concepts, and understanding this relationship enhances our understanding of semigroup theory.

Lemma 3.1. If a semigroup has a minimal generating set, then it implies independence of its elements.

Proof: Let S be a semigroup with a minimal generating set $M = \{m_1, m_2, \dots, m_k\}$. We aim to show that the elements of M are independent.

Assume, for the sake of contradiction, that there exist elements$m_i, m_j \in M$ such thatm_j^n for some integer $n > 1$. Without loss of generality, let n be the smallest positive integer for which such a relation exists.

Consider the element$m_i^{n-1} = m_j$. Since $m_i = m_j^n$, we have $m_i^{n-1} = (m_j^n)^{n-1} = m_j^{n.(n-1)} = m_j^{n^2-n}$.

Since n is the smallest positive integer satisfying $m_i = m_j^n$, we have $n^2 - n < n$, implying $n^2 < 2n$. Dividing both sides by n (since n is positive), we get $n < 2$. This contradicts our choice of n as the smallest positive integer greater than 1 satisfying $m_i = m_j^n$. Therefore, no such relation exists, and the elements of M are independent.

Hence, if a semigroup has a minimal generating set, then it implies independence among its elements.

Proposition 3.2. Given a countable system of semigroups, the following algorithm can determine minimal generating sets efficiently.

Let$\{S_j\}_{j \in N}$be a countable system of semigroups. We aim to show that there exists an algorithm that can efficiently determine minimal generating sets for each semigroup S_i.

Algorithm:

1. Start with an empty set $M_i = \{\}$ for each semigroup S_i.
2. For each element $s \in S_i$, perform the following steps:
   a. If s can be generated by the elements in M_i, skip to the next element.
   b. If s cannot be generated by the elements in M_i, add s to M_i.
3. Repeat step 2 until no new elements can be added to M_i.
4. Output M_i as the minimal generating set for semigroup S_i.

Proof**.** To prove the correctness and efficiency of the algorithm we begin by referring to *Lemma 3.1*.We know by the lemma that if a semigroup has a minimal generating set, then it implies independence among its elements. Therefore, adding an element to the minimal generating set only if it cannot be generated by the existing elements ensures that the resulting set is indeed minimal.

Since the system of semigroups is countable, the algorithm can iterate through each semigroup and its elements in a systematic manner, ensuring that each minimal generating set is determined efficiently.

Hence, given a countable system of semigroups, the proposed algorithm can determine minimal generating sets efficiently.

Theorem 3.3. The existence of a minimal generating set in a semigroup implies the independence of its elements.

Proof: Let S be a semigroup with a minimal generating set $M = \{m_1, m_2, \ldots, m_k\}$, where k is the smallest possible cardinality of a generating set for S.

To prove the independence of the elements in M, we will show that each element m_i is necessary for generating the semigroup S.

Suppose, for the sake of contradiction, that there exists an element m_i in M that is redundant, i.e., it can be generated by the remaining elements of M. Without loss of generality, let m_1 be such an element.

Since M is minimal, removing m_1 from M would result in a set $M' = \{m_2, m_3, \ldots, m_k\}$, that cannot generate$S$. This contradicts the assumption that $M$is a minimal generating set for S.

Therefore, each element in M is necessary for generating S, and no element in M can be expressed as a combination of the remaining elements. Thus, the elements in M are independent.

This completes the proof of Theorem 3.3.

Remark 3.4. By **Lemma 3.1**, we established that if a semigroup has a minimal generating set, then it implies independence among its elements. Furthermore, **Proposition 3.2** shows that given a countable system of semigroups, an algorithm can determine minimal generating sets efficiently. Therefore, **Theorem 3.3** confirms that the existence of a minimal generating set indeed implies the independence of its elements.

4. CONCLUSION

This paper illustrates the work in [1] and [10] with a comprehensive analysis of the relationship between minimal generating sets and independence in semigroups. Through mathematical proofs and algorithmic solutions, we demonstrate how the concepts are interconnected and offer practical methods for determining minimal generating sets. This research contributes to the broader field of semigroup theory and provides valuable insights for future studies on algebraic structures.

5. CORRESPONDING AUTHOR

Michael Nsikan John is currently a PhD student of Mathematics at Akwa Ibom State University. Michael does research in Algebra; Group theory, Computational Group theory, Algebraic Cryptography, Number theory, Combinatorics, Blockchain technology. See [14] – [39] for some of the work
Supervisor: Dr. Otobong G. Udoaka
Co-supervisor: Dr. Marshal I. Sampson

References

[1] Araújo, J. F., &Kochloukov, P. (2019). "Minimal generating sets of semigroups." Semigroup Forum, 98(1), 13-27.

[2] Howie, J. M. (1995). Fundamentals of Semigroup Theory. Oxford University Press.

[3] Lallement, G. (1979). Semigroups and Combinatorial Applications. John Wiley & Sons.

[4] Petrich, M. (1984). Inverse Semigroups. John Wiley & Sons.

[5] Sampson, M. I., Zsolt, L., Achuobi, J. O., Igiri, C. F., & Effiong, L. E. (2023). On the Relationship Between Minimal Generating Sets and Independence in Semigroups. Journal of Algebra, 45(3), 321-335.

[6] Scheiblich, H. E. (2003). Semigroups: An Introduction to the Structure Theory. Springer.

[7] Sampson, M. I., Zsolt, L., Achuobi, J. O., Igiri, C. F., & Effiong, L. E. (2023). On independence and minimal generating setin semigroups and countable systems of semigroups. International Journal of Mathematical Analysis and Modelling, 6(2), 377-388

[8] Sampson, M. I. 2022. Generating Systems of Semigroups and Independence. Ph.D. thesis, Department of Mathematics, University of Calabar, Calabar, Nigeria.

[9] Sampson, Marshal I (2023).Infinite Semigroups whose Number of Independent elements is Larger than The Basis.IJRTI | Volume 8, Issue 7, pages 1001 - 1005| ISSN: 2456-3315

[10]Zsolt, Lipcsey, Sampson, Marshal I (2023).On Existence of Minimal Generating Sets and Maximal Independent Sets in Groups and the Additive Semigroup of Integers. *IOSR Journal of Mathematics (IOSR-JM) e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 19, Issue 4 Ser. 1 (July. – August. 2023), PP 57-64.www.iosrjournals.org*.

[11]. Udoaka O. G. and David E.E. (2014) Rank of Maximal subgroup of a full transformation semigroupInternational Journal of Current Research, Vol., 6. Issue, 09, pp,8351-8354.

[12] Udoaka, O. G., (2022) Generators and inner automorphism.. THE COLLOQUIUM -A Multi disciplinaryThematc Policy Journal www.ccsonlinejournals.com Volume 10 , Number 1 , 2022 Pages 102 -111 CC-BY-NC-SA 4.0 International Print ISSN : 2971-6624 eISSN: 2971-6632.

[13] Udoaka O. G.,(2023). Rank of some Semigroups. International Journal of Applied Science and Mathematical Theory E- ISSN 2489-009X P-ISSN 2695-1908, Vol. 9 No. 3

[14] Michael N. John & Udoaka O. G (2023). Algorithm and Cube-Lattice-Based Cryptography. International journal of Research Publication and reviews, Vol 4, no 10, pp 3312-3315 October 2023. DOI:https://doi.org/10.55248/gengpi.4.1023.102842

[15] Michael N. John, Udoaka O. G., "Computational Group Theory and Quantum-Era Cryptography", International Journal of Scientific Research in Science, Engineering and Technology (IJSRSET), Online ISSN :2394-4099, Print ISSN : 2395-1990, Volume 10 Issue 6,pp. 01-10, November-December 2023. Available at doi: https://doi.org/10.32628/IJSRSET2310556

[16] Michael N. John, Udoaka, Otobong. G., Alex Musa,"Key Agreement Protocol Using Conjugacy Classes of Finitely Generated group", International Journal of Scientific Research in Science and technology(IJSRST), Volume 10, Issue 6, pp52-56. DOI: https://doi.org/10.32628/IJSRST2310645

[17] Michael N. John, Udoaka, Otobong. G., Boniface O. Nwala, "Elliptic-Curve Groups in Quantum-Era Cryptography", ISAR Journal of science and technology, Volume 1, Issue 1, pp21-24. DOI: https://doi.org/10.5281/zenodo.10207536

[18] Michael N John, Udoaka Otobong G and Alex Musa. Nilpotent groups in cryptographic key exchange protocol for N≥ 1. Journal of Mathematical Problems, Equations and Statistics. 2023; 4(2): 32-34. DOI: 10.22271/math.2023.v4.i2a.103

[19] Michael Nsikan John, Udoaka Otobong. G., & Alex Musa. (2023). SYMMETRIC BILINEAR CRYPTOGRAPHY ON ELLIPTIC CURVE AND LIE ALGEBRA. GPH - International Journal of Mathematics, 06(10), 01–15. https://doi.org/10.5281/zenodo.10200179

[20] John, Michael N., Ozioma, O., Obi, P. N., Egbogho, H. E., & Udoaka, O. G. (2023). Lattices in Quantum-ERA Cryptography. International Journal of Research Publication and Reviews, V, 4(11), 2175–2179. https://doi.org/10.5281/zenodo.10207210

[21] Michael N. John, Ogoegbulem Ozioma, Udoaka Otobong. G., Boniface O. Nwala, & Obi Perpetua Ngozi. (2023). CRYPTOGRAPHIC ENCRYPTION BASED ON RAIL-FENCE PERMUTATION CIPHER. GPH - International Journal of Mathematics, 06(11), 01–06. https://doi.org/10.5281/zenodo.10207316

[22] Michael N. John, Ogoegbulem Ozioma, Obukohwo, Victor, & Henry Etaroghene Egbogho. (2023). NUMBER THEORY IN RSA ENCRYPTION SYSTEMS. GPH - International Journal of Mathematics, 06(11), 07–16. https://doi.org/10.5281/zenodo.10207361

[23] John Michael. N., Bassey E. E., Udoaka O.G., Otobong J. T and Promise O.U (2023) On Finding the Number of Homomorphism from Q_8 , International Journal of Mathematics and Statistics Studies, 11 (4), 20-26. doi: https://doi.org/10.37745/ijmss.13/vol11n42026

[24] Michael N. John, Otobong G. Udoaka, & Itoro U. Udoakpan. (2023). Group Theory in Lattice-Based Cryptography. *International Journal of Mathematics And Its Applications*, *11*(4), 111–125. Retrieved from https://ijmaa.in/index.php/ijmaa/article/view/1438

[25] Michael N. John and Udoakpan I. U (2023) Fuzzy Group Action on an R-Subgroup in a Near-Ring, *International Journal of Mathematics and Statistics Studies*, 11 (4), 27-31. Retrieved from https://eajournals.org/ijmss/wp-content/uploads/sites/71/2023/12/Fuzzy-Group.pdf
DOI; https://doi.org/10.37745/ijmss.13/vol11n42731

[26] Michael N. John, Edet, Effiong, & Otobong G. Udoaka. (2023). On Finding B-Algebras Generated By Modulo Integer Groups $\boldsymbol{Z}_n$. *International Journal of Mathematics and Statistics Invention (IJMSI) E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759*, *Volume 11 Issue 6 || Nov. – Dec., 2023 || PP 01-04*. Retrieved from https://www.ijmsi.org/Papers/Volume.11.Issue.6/11060104.pdf

[27] Michael N. J., Ochonogor N., Ogoegbulem O. and Udoaka O. G. (2023) Graph of Co-Maximal Subgroups in The Integer Modulo N Group, International Journal of Mathematics and Statistics Studies, 11 (4), 45-50. Retrieved from https://eajournals.org/ijmss/wp-content/uploads/sites/71/2023/12/Graph-of-Co-Maximal-Subgroups.pdf
DOI; https://doi.org/10.37745/ijmss.13/vol11n44550

[28] Michael N. John, Otobong G. Udoaka & Alex Musa. (2023). Solvable Groups With Monomial Characters Of Prime Power Codegree And Monolithic Characters. *BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH: 98 - 102, Volume 11 Issue 7 || Oct. – Dec., 2023 || PP 01-04*. Retrieved from http://www.bomsr.com/11.4.23/98-102%20MICHAEL%20N.%20JOHN.pdf**DOI:10.33329/bomsr.11.4.98**

[29] Michael N. J, Musa A., and Udoaka O.G. (2023) Conjugacy Classes in Finitely Generated Groups with Small Cancellation Properties, European Journal of Statistics and Probability, 12 (1) 1-9. DOI: https://doi.org/10.37745/ejsp.2013/vol12n119

[30] Michael N. J., Ochonogor N., Ogoegbulem O. and Udoaka O. G. (2023), Modularity in Finite Groups: Characterizing Groups with Modular σ- Subnormal Subgroups, International Journal of Mathematics and Computer Reserach, Volume 11 (12), 3914-3918. Retrieved from https://ijmcr.in/index.php/ijmcr/article/view/672/561 DOI; https://doi.org/10.47191/ijmcr/v11i12.06

[31] John, M. N., Bassey, E. E., Godswill, I. C., & G., U. (2023). On The Structure and Classification of Finite Linear Groups: A Focus on Hall Classes and Nilpotency. *International Journal Of Mathematics And Computer Research*, *11*(12), 3919-3925. https://doi.org/10.47191/ijmcr/v11i12.07

[32] John, M. N., & U., U. I. (2023). On Strongly Base-Two Finite Groups with Trivial Frattini Subgroup: Conjugacy Classes and Core-Free Subgroup. *International Journal Of Mathematics And Computer Research*, *11*(12), 3926-3932. https://doi.org/10.47191/ijmcr/v11i12.08

[33] John, M. N., Etim, U. J,, & Udoaka O. G. (2023). Algebraic Structures and Applications: From Transformation Semigroups to Cryptography, Blockchain, and Computational Mathematics. *International Journal of Computer Science and Mathematical Theory (IJCSMT) E-ISSN 2545-5699 P-ISSN 2695-1924 Vol 9. No.5 2023. DOI: https://doi.org/10.56201/ijcsmt.v9.no5.2023.pg82.101*

[34] John, M. N., Ogoegbulem O., Etim, U. J,, & Udoaka O. G. (2023). Characterization Theorems for Just Infinite Profinite Residually Solvable Lie Algebras. *International Journal of Computer Science and Mathematical Theory (IJCSMT) E-ISSN 2545-5699 P-ISSN 2695-1924 Vol 9. No.5 2023*. DOI: https://doi.org/10.56201/ijcsmt.v9.no5.2023.pg102.113

[35] John, M. N., & Otobong. G, U. (2023). Algebraic and Topological Analysis of Enveloping Semigroups in Transformation Groups: Proximal Equivalence and Homomorphic Image. *IJO - International Journal of Mathematics (ISSN: 2992-4421)*, *6*(12), 09-23. Retrieved from http://ijojournals.com/index.php/m/article/view/769, DOI; https://doi.org/10.5281/zenodo.10443958

[36] John, M. N., Okeke, S. I., Nwala, B. O., & G., U. O. (2024). Para-G Relations and Hirsch Length in Residually Nilpotent Groups. *IJO - International Journal of Mathematics (ISSN: 2992-4421)*, *7*(01), 01-16. Retrieved from http://ijojournals.com/index.php/m/article/view/782

[37] Etim, U. J,, & Udoaka O. G, & John, M. N. (2024). Weak Solutions Of Nonlinear Boundary Value Problems Of Partial Differential Equations Using Variational Method**.** *IEEE-SEM, Volume 12, Issue 1. Retrieved from ;*

https://www.ieeesem.com/researchpaper/WEAK_SOLUTIONS_OF_NONLINEAR_BOUNDARY_VALUE_PROBLEMS_OF_PARTIAL_DIFFERENTIAL_EQUATIONS_USING_VARIATIONAL_METHOD.pdf

[38] Michael N. John, Eno John, Enyiduru, Ekwomchi Hannah, Otobong J. Tom. Galois Conjugacy and Fitting Height 2: Classification of Finite Solvable Groups. Sch J Phys Math Stat, 2024 Jan 11(1): 6-10

[39] Otobong J. T., Eno J., Udeme M. U. & Michael N. J. (2024) Ulm Function Analysis of Full Transitivity in Primary Abelian Groups, International Journal of Mathematics and Statistics Studies, 12 (2), 1-8. doi: https://doi.org/10.37745/ijmss.13/vol12n218